ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE EQUATION OF HIGH-INTENSITY HEAT EXCHANGE

A. V. Finkel'shtein

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The convergence of solutions of the equation of high-intensity heat exchange to solutions of the classical problem of heat conduction is established when the relaxation time of the thermal process tends to zero.

Effects of high-intensity heat exchange are described by equations of the form [1]

$$q+\varepsilon\frac{\partial q}{\partial\tau}=-\Lambda\frac{\partial T}{\partial x}, \tag{1}$$

$$\frac{\partial T}{\partial\tau}+\varepsilon\frac{\partial^2 T}{\partial\tau^2}=\mathrm{Fo}\frac{\partial^2 T}{\partial x^2}, \tag{2}$$

where $\varepsilon = \tau_r/\tau_0$ is a small quantity. Therefore, it is natural to consider system (1), (2) as a system with a small parameter, and use asymptotic methods [2-5] for its analysis and solution. Equations (1), (2) with $\varepsilon = 0$ will be called unperturbed, while for $\varepsilon > 0$ they are perturbed. A quite general form of expanding solutions of Eq. (2) in powers of the small parameter ε was obtained in [6]. This expansion can be used to determine solutions of the hyperbolic equation of heat conduction. Below we construct such an expansion, making it possible to establish convergence of solutions of the perturbed equations to solutions of the unperturbed ones for $\varepsilon \to 0$.

We denote by $q_0(x, \tau)$, $T_0(x, \tau)$ the solutions of the unperturbed equations. Consider the heat-conduction process in a layer of material of thickness l. We supplement (1), (2) by the initial conditions

$$q(x,\ 0)=\varkappa(x), \tag{3}$$

$$T(x,\ 0)=\theta_0(x),\ \frac{\partial T(x,\ 0)}{\partial\tau}=\theta_1(x). \tag{4}$$

As boundary conditions we select conditions of type I:

$$T(0,\ \tau)=\varphi_1(\tau),\ T(l,\ \tau)=\varphi_2(\tau). \tag{5}$$

The solution of the boundary-value problem (2), (4), (5) is represented in the form

$$T(x,\ \tau,\ \varepsilon)=T_0(x,\ \tau)+\varepsilon T_1(x,\ \tau)+(\varepsilon\pi_1(x)+\varepsilon^2\pi_2(x))\exp\left(-\frac{\tau}{\varepsilon}\right)+u(x,\ \tau,\ \varepsilon), \tag{6}$$

where the functions $T_0(x, \tau)$, $T_1(x, \tau)$, $\pi_1(x)$, $\pi_2(x)$, $u(x, \tau, \varepsilon)$ satisfy the following boundary-value problems:

$$\frac{\partial T_0}{\partial\tau}=\mathrm{Fo}\frac{\partial^2 T_0}{\partial x^2},\ T_0(x,\ 0)=\theta_0(x),\ T_0(0,\ \tau)=\varphi_1(\tau),\ T_0(l,\ \tau)=\varphi_2(\tau); \tag{7}$$

$$\frac{\partial T_1}{\partial\tau}=\mathrm{Fo}\frac{\partial^2 T_1}{\partial x^2}-\frac{\partial^2 T_0}{\partial\tau^2},\ T_1(x,\ 0)=-\pi_1(x),\ T_1(0,\ \tau)=T_1(l,\ \tau)=0; \tag{8}$$

$$\pi_1(x)=\frac{\partial T_0(x,\ 0)}{\partial\tau}-\theta_1(x),\quad \pi_2(x)=\frac{\partial T_1(x,\ 0)}{\partial\tau},\ \pi_1''(0)=\pi_1(l)=\pi_2''(0)=\pi_2''(l)=0; \tag{9}$$

$$\frac{\partial u}{\partial\tau}+\varepsilon\frac{\partial^2 u}{\partial\tau^2}=\mathrm{Fo}\frac{\partial^2 u}{\partial x^2}+\mathrm{Fo}(\varepsilon\pi_1''(x)+\varepsilon^2\pi_2''(x))\exp\left(-\frac{\tau}{\varepsilon}\right)-\varepsilon^2\frac{\partial^2 T_1(x,\ \tau)}{\partial\tau^2}, \tag{10}$$

$$u(x,\ 0,\ \varepsilon)=\frac{\partial u(x,\ 0,\ \varepsilon)}{\partial\tau}=0,\ u(0,\ \tau,\ \varepsilon)=u(l,\ \tau,\ \varepsilon)=0.$$

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We allow that the functions $\pi_i''(x)$ and $\partial^2T_1/\partial\tau^2$ be decomposed into uniformly (and absolutely) converging Fourier series:

$$\pi_i''(x) = \sum_{n=1}^{\infty} \pi_{in} \sin \frac{n\pi x}{l}; \quad \frac{\partial^2 T_1(x, \tau)}{\partial\tau^2} = \sum_{n=1}^{\infty} \eta_n(\tau) \sin \frac{n\pi x}{l}, \tag{11}$$

whose coefficients are determined by the well-known equations of [7].

The function $u(x, \tau, \varepsilon)$ will be sought in the form of the series

$$u(x, \tau, \varepsilon) = \sum_{n=1}^{\infty} u_n(\tau, \varepsilon) \sin \frac{n\pi x}{l}. \tag{12}$$

We substitute (11), (12), into (10), obtaining a system of boundary-value problems to determine the functions $u_n(\tau, \varepsilon)$, $n = 1, 2, \ldots$:

$$\varepsilon \ddot{u}_n + \dot{u}_n + \left(\frac{n\pi}{l}\right)^2 \mathrm{Fo}\, u_n = p_n(\tau, \varepsilon), \quad \dot{u}_n(0, \varepsilon) = u_n(0, \varepsilon) = 0, \tag{13}$$

$$p_n(\tau, \varepsilon) = \mathrm{Fo} \exp\left(-\frac{\tau}{\varepsilon}\right) \sum_{i=1}^{2} \varepsilon^i \pi_{in} - \varepsilon^2 \eta_n(\tau). \tag{14}$$

We denote the roots of the characteristic equation of problem (13) by k_1 and k_2:

$$k_1 = -(1 + \Delta_n^{1/2})/2\varepsilon, \quad k_2 = -(1 - \Delta_n^{1/2})/2\varepsilon, \quad \Delta_n = 1 - 4\mathrm{Fo}\varepsilon (n\pi/l)^2. \tag{15}$$

Solving problem (13) by the method of variation of arbitrary constants, we obtain

$$u_n(\tau,\varepsilon) = \begin{cases} \int_0^\tau p_n(s, \varepsilon) \Delta_n^{-1/2} (\exp[k_2(\tau - s)] - \exp[k_1(\tau - s)])\, ds \text{ for } \Delta_n > 0, \\ \int_0^\tau p_n(s, \varepsilon) \frac{(\tau - s)}{\varepsilon} \exp\left(-\frac{\tau - s}{2\varepsilon}\right) ds \text{ for } \Delta_n = 0, \\ \int_0^\tau p_n(s, \varepsilon)\, 2|\Delta_n|^{-1/2} \exp\left(-\frac{\tau - s}{2\varepsilon}\right) \sin \frac{|\Delta_n|^{1/2}(\tau - s)}{2\varepsilon}\, ds \text{ for} \\ \Delta_n < 0. \end{cases} \tag{16}$$

Substituting expression (14) into (16) and evaluating the integrals obtained, we have

$$u_n(\tau, \varepsilon) = 2\,\mathrm{Fo}\, a_n(\tau, \varepsilon) \sum_{i=1}^{2} \varepsilon^{i+1} \pi_{in} - \varepsilon^2 \int_0^\tau b_n(\tau, \varepsilon, s) \eta_n(s)\, ds, \tag{17}$$

where

$$a_n(\tau,\varepsilon) = \begin{cases} (1 - \Delta_n)^{-1} \left\{ \left(\exp\left[-\frac{(1 + \Delta_n^{1/2})\tau}{2\varepsilon}\right] - \exp\left[-\frac{(1 - \Delta_n^{1/2})\tau}{2\varepsilon}\right] \right) \times \right. \\ \times \Delta_n^{-1/2} + \exp\left[-\frac{(1 + \Delta_n^{1/2})\tau}{2\varepsilon}\right] + \exp\left[-\frac{(1 - \Delta_n^{1/2})\tau}{2\varepsilon}\right] - \\ \left. - 2\exp\left(-\frac{\tau}{\varepsilon}\right) \right\}, \quad \Delta_n > 0; \\ \frac{1}{\varepsilon} \exp\left(-\frac{\tau}{2\varepsilon}\right) \left[1 - 2\varepsilon + 2\varepsilon \exp\left(-\frac{\tau}{2\varepsilon}\right)\right], \quad \Delta_n = 0; \\ 2(1 + |\Delta_n|)^{-1} \exp\left(-\frac{\tau}{2\varepsilon}\right) \left[|\Delta_n|^{-1/2} \sin \frac{|\Delta_n|^{1/2}\tau}{2\varepsilon} - \right. \\ \left. - \cos \frac{|\Delta_n|^{1/2}\tau}{2\varepsilon} + \exp\left(-\frac{\tau}{2\varepsilon}\right) \right], \quad \Delta_n < 0; \end{cases} \tag{18}$$

$$b_n(\tau, \varepsilon, s) = \begin{cases} \Delta_n^{-1/2}\left(\exp\left[-\frac{(1-\Delta_n^{1/2})(\tau-s)}{2\varepsilon}\right] - \exp\left[-\frac{(1+\Delta_n^{1/2})(\tau-s)}{2\varepsilon}\right]\right), & \Delta_n > 0; \\ \frac{(\tau-s)}{\varepsilon}\exp\left(-\frac{\tau-s}{2\varepsilon}\right), & \Delta_n = 0; \\ 2|\Delta_n|^{-1/2}\exp\left(-\frac{\tau-s}{2\varepsilon}\right)\sin\frac{|\Delta_n|^{1/2}(\tau-s)}{2\varepsilon}, & \Delta_n < 0. \end{cases} \quad (19)$$

Thus, the following representation is valid for the function $u(x, \tau, \varepsilon)$

$$u(x, \tau, \varepsilon) = 2\,\mathrm{Fo}\sum_{i=1}^{2}\varepsilon^{i+1}\sum_{n=1}^{\infty}a_n(\tau, \varepsilon)\pi_{in}\sin\frac{n\pi x}{l} - \varepsilon^2\sum_{n=1}^{\infty}\left(\int_0^{\tau}b_n(\tau, \varepsilon, s)\eta_n(s)\,ds\right)\sin\frac{n\pi x}{l}. \quad (20)$$

It follows from Eqs. (18), (19) that for $\tau > 0$ and sufficiently small $\varepsilon \geqslant 0$ we have for all $n = 1, 2, \ldots$

$$|a_n(\tau, \varepsilon)| \leqslant C_1, \quad |b_n(\tau, \varepsilon)| \leqslant C_1. \quad (21)$$

Indeed, Eqs. (18), (19) show that one must investigate the boundedness of the functions $a_n(\tau, \varepsilon)$ and $b_n(\tau, \varepsilon)$ at $\Delta_n \to 0$ and $\Delta_n \to 1$. Since $\lim_{\Delta_n^{1/2}\to 0}[(\exp(-\Delta_n^{1/2}\tau/2\varepsilon) - \exp(\Delta_n^{1/2}\tau/2\varepsilon))/\Delta_n^{1/2}] = -\tau/\varepsilon$, one can select the number $\delta_1 > 0$ in such a manner that $|(\exp(-\Delta_n^{1/2}\tau/2\varepsilon) - \exp(\Delta_n^{1/2}\tau/2\varepsilon))/\Delta_n^{1/2}| \leqslant \tau/\varepsilon + 0.1$ as only $0 < \Delta_n^{1/2} < \delta_1$. From Eq. (18) we obtain for $0 < \Delta_n^{1/2} < \delta_1$

$$|a_n(\tau, \varepsilon)| < \frac{1}{1-\delta_1^2}\left[\exp(-\tau/2\varepsilon)\left(\frac{\tau}{\varepsilon} + 0.1\right) + \exp(-(1-\delta_1)\tau/2\varepsilon) + \exp(-\tau/2\varepsilon) + 2\exp(-\tau/\varepsilon)\right]. \quad (22)$$

Consider $a_n(\tau, \varepsilon)$ as $\Delta_n \to 1$. We transform Eq. (18):

$$a_n(\tau, \varepsilon) = \Delta_n^{1/2}\left[\frac{\exp(-\tau/2\varepsilon)(\exp(-\Delta_n^{1/2}\tau/2\varepsilon) - \exp(-\tau/2\varepsilon))}{1-\Delta_n^{1/2}} - \frac{\exp(-(1-\Delta_n^{1/2})\tau/2\varepsilon) - \exp(-\tau/\varepsilon)}{1+\Delta_n^{1/2}}\right]. \quad (23)$$

Since $\lim_{1-\Delta_n^{1/2}\to 0}[(\exp(-\Delta_n^{1/2}\tau/2\varepsilon) - \exp(-\tau/2\varepsilon))/(1-\Delta_n^{1/2})] = \frac{\tau}{2\varepsilon}$, one can select a number $\delta_2 > 0$ so that $|(\exp(-\Delta_n^{1/2}\tau/2\varepsilon) - \exp(-\tau/2\varepsilon))/(1-\Delta_n^{1/2})| \leqslant \frac{\tau}{2\varepsilon} + 0.1$, as only $1 - \Delta_n^{1/2} < \delta_2$. We note that for $\Delta_n \to 1$ we have, due to (15), $1 - \Delta_n^{1/2} = 2\mathrm{Fo}\varepsilon(\pi n/l)^2 > 2\mathrm{Fo}\varepsilon(\pi/l)^2$. From Eq. (23) we obtain for $1 - \Delta_n^{1/2} < \delta_2$

$$|a_n(\tau, \varepsilon)| \leqslant \frac{1}{1-\delta_2}\left[\exp(-\tau/2\varepsilon)(\tau/2\varepsilon + 0.1) + \frac{1}{2-\delta_2}(\exp(-\mathrm{Fo}(\pi/l)^2\tau) + \exp(-\tau/\varepsilon))\right]. \quad (24)$$

One similarly estimates $a_n(\tau, \varepsilon)$ for $\Delta_n < 0$. Estimates (22), (24) show that if ε is selected to be sufficiently small, so that $(\tau/\varepsilon)\exp(-\tau/2\varepsilon) < \delta$, then estimate (21) is valid for $a_n(\tau, \varepsilon)$. The functions $b_n(\tau, \varepsilon)$ are estimated similarly.

Due to the absolute convergence of series (11)

$$\sum_{n=1}^{\infty}|\pi_{in}| \leqslant C_2, \; i = 1,2, \quad \sum_{n=1}^{\infty}|\eta_n(\tau)| \leqslant C_2. \quad (25)$$

Expression (20) with account of (21), (25) lead to the estimate $|u(x, \tau, \varepsilon)| \leqslant C \times (2\,\mathrm{Fo}(\varepsilon^2 + \varepsilon^3) + \tau\varepsilon^2)$. Consequently, for finite time values

$$u(x, \tau, \varepsilon) = O(\varepsilon^2). \quad (26)$$

Expression (26) shows that the representation of the solutions of the boundary-value problem (2), (4), (5) in form (6)-(10) makes it possible to investigate for $\varepsilon \to 0$ the asymptotic behavior of the solutions of the hyperbolic equation of heat conduction accurately up to terms in ε^2. It follows from Eq. (6) that for $T_1(x, \tau)$, $\pi_i(x)$ and finite τ values the solution of the hyperbolic equation of heat conduction tends to the solution of the classical parabolic problem uniformly in (x, τ), $\tau \geqslant 0$, $0 \leqslant x \leqslant l$.

We note that the function $\partial u/\partial x$ satisfies the system (10), whose right-hand side consists of the functions $\pi_i^{!''}(x)$ and $\partial^2(\partial T_1/\partial x)/\partial\tau^2$. Thus, if expansions of the form (11) occur for the latter functions, then an equation similar to (26) is valid for $\partial u/\partial x$. Consequently, for $\varepsilon \to 0$

$$\frac{\partial T(x, \tau, \varepsilon)}{\partial x} \to \frac{\partial T_0(x, \tau)}{\partial x}. \tag{27}$$

We write down the solution of problem (1), (3):

$$q(x, \tau, \varepsilon) = \varkappa(x)\exp\left(-\frac{\tau}{\varepsilon}\right) + \frac{1}{\varepsilon}\int_0^\tau \exp\left(-\frac{\tau - s}{\varepsilon}\right)\left(-\Lambda\frac{\partial T(x, s, \varepsilon)}{\partial x}\right)ds.$$

We assume that the temperature gradient $\partial T(x, \tau, \varepsilon)/dx$ is a continuously differentiable function in τ. Using then integration by parts, we obtain:

$$q(x, \tau, \varepsilon) = -\Lambda\frac{\partial T(x, \tau, \varepsilon)}{\partial x} + \left(\varkappa(x) + \Lambda\frac{\partial T(x, 0, \varepsilon)}{\partial x}\exp\left(-\frac{\tau}{\varepsilon}\right)\right) + \int_0^\tau \exp\left(-\frac{\tau - s}{\varepsilon}\right)\frac{\partial}{\partial s}\left(\Lambda\frac{\partial T(x, s, \varepsilon)}{\partial x}\right)ds. \tag{28}$$

From Eq. (28) with account of (27) and the definition of $q_0(x, \tau)$ follows then for $\varepsilon \to 0$ the pointwise convergence $q(x, \tau, \varepsilon) \to q_0(x, \tau)$ if $\tau > 0$. This convergence is uniform in (x, τ) for $\tau \geqslant 0$, $0 \leqslant x \leqslant l$ if $\varkappa(x) = -\Lambda\dfrac{\partial T(x, 0)}{\partial x}$.

Thus, under conditions that the original data of the problem guarantee the required smoothness of the functions $\pi_i(x)$ and $T_1(x, \tau)$, the solutions of the equation of high-intensity heat exchange tend to the solutions of the classical problem of heat conduction when the relaxation time of the thermal process tends to zero.

NOTATION

T, temperature; q, heat flux; x, a spatial coordinate; τ, time; Λ, dimensionless thermal conductivity, $\Lambda = \lambda T^*/lq^*$; λ, thermal conductivity; T^* and q^*, temperature and heat flux scales; Fo, Fourier number $Fo = a\tau_0/l^2$; a, thermal diffusivity; τ_0, time scale; τ_r, relaxation time of the thermal process, and when the sign / is used in equations it is assumed that division is performed over all quantities appearing after this sign.

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